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Dynamical squeezing in quantum mechanics

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Abstract. The time evolution orbits of Gaussian vectors under a harmonic oscillator dynamics are computed and discussed. The generic case is shown to be this: a minimum uncertainty vector, which is coherent with respect to a harmonic oscillator Hamiltonian with smaller frequency than the one of the driving Hamiltonian, evolves into a minimum uncertainty vector with reduced position variance for exactly two discrete instants of time per period. The results are applied to the case of a perturbing frequency jump in the driving harmonic oscillator Hamiltonian. The total transition probability to the excited (unperturbed) oscillator levels from the ground-state vector is computed.

1. Introduction and conclusions

Wavepackets, whose momentum and position variance saturate Heisenberg's uncertainty relation, i.e. obey $(\Delta P)_\Phi(\Delta Q)_\Phi = \hbar/2$, were introduced in the early days of quantum mechanics in order to simulate classical conditions as closely as possible [1]. Meanwhile among these 'quasiclassical' vectors the subsets of those with identical frequency parameters have gained more practical importance as the 'coherent states' of the electromagnetic single-mode radiation field [2]. Recently a class of time evolutions has been constructed [3], which evolve for all times every quasiclassical vector into a quasiclassical one and decrease its position variance. Completely independent from any dynamics, a quasiclassical vector, whose position variance is less than the position variance of a certain quasiclassical reference vector, is called 'squeezed' with respect to the reference vector [4-8]. (In the context of quantum optics the vacuum vector is taken as the universal reference standard.) The time evolutions presented in [3] can thus be said to induce 'dynamical squeezing' in the sense that the later of two vectors on a curve, which is generated through the time evolution of a quasiclassical initial vector, is squeezed relative to the earlier one. The way this goal is achieved in [3] is as follows. Take as the family of time evolution operators from time 0 to time t an arbitrary (differentiable) curve $U[\mu(t)]$ through the one-parameter Lie group of unitary Bogoliubov transformations, which are defined by

$$U[\mu]QU[\mu]^\dagger = \mu^{-1}Q \qquad U[\mu]PU[\mu]^\dagger = \mu P.$$

The function $\mu(t)$ is chosen differentiable, increasing and strictly positive with $\mu(0) = 1$. Obviously, position variances decrease. Quasiclassical vectors remain quasiclassical for all times under a time evolution of this kind, since the product $(\Delta P)_{\Phi(t)}(\Delta Q)_{\Phi(t)}$ is constant in t for all initial vectors $\Phi(0)$ with well defined P and Q variances $(\Phi(t) := U[\mu(t)]\Phi(0))$. (The interesting problem of how such a squeezing dynamics can be realised in the laboratory seems to be unresolved.)

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Due to the fact that the squeezing Bogoliubov transformations given above are not generated by a harmonic oscillator Hamiltonian the authors of [3] deny the very possibility of inducing dynamical squeezing through a harmonic oscillator dynamics (with a generally time-dependent frequency parameter). Obviously, the previously described curves passing through the set of squeezing Bogoliubov transformations lead to decreasing $(\Delta Q)_{\Phi(t)}$ not only for all quasiclassical initial vectors but for all vectors in the Hilbert space (for which $(\Delta Q)_{\Phi(0)}$ is finite). This, however, is an unnecessarily restricted way of inducing squeezing dynamically, as the notion of squeezing refers to pairs of quasiclassical vectors only. There is no need even to have all quasiclassical initial vectors squeezed by one and the same dynamics.

In this paper I relax the too-rigid notion of dynamical squeezing of [3] and I prove that a harmonic oscillator dynamics induces squeezing on a set of quasiclassical vectors, though not on all. The phenomenon does not take place at all times but at certain discrete ones. In order to see this, the action of an oscillator dynamics on the set of Gaussian vectors (as defined precisely in definition 2.1) is computed explicitly. The set of Gaussian vectors contains the quasiclassical ones as a subset and is invariant under the harmonic oscillator time evolution. Thus closed curves are generated within the set of Gaussian vectors. These orbits are given in proposition 4.1. It turns out that every orbit either remains for all times within the set of vectors, which are coherent with respect to the driving dynamics, or else passes during one oscillator period $T=2\pi/\omega$ exactly four times through the set of quasiclassical vectors. For a curve of the latter type the time dependence of the momentum and position variances along the orbit is given in example 4.1. If the initial vector at $t=0$ is the ground-state vector of a harmonic oscillator Hamiltonian with frequency ω', which can be assumed without loss of generality, and if the time evolution is generated by the harmonic oscillator Hamiltonian with the frequency $\omega>\omega'$ (and the same mass parameter), then $(\Delta P)_t$ and $(\Delta Q)_t$ oscillate with a period π/ω between the following bounds:

$$(\Delta Q)^2_{\pi/2\omega}=\frac{\hbar}{2m\omega'}\left(\frac{\omega'}{\omega}\right)^2\leqslant(\Delta Q)^2_t\leqslant\frac{\hbar}{2m\omega'}=(\Delta Q)^2_0 \tag{1.1}$$

$$(\Delta P)^2_0=\frac{\hbar m\omega'}{2}\leqslant(\Delta P)^2_t\leqslant\frac{\hbar m\omega'}{2}\left(\frac{\omega}{\omega'}\right)^2=(\Delta P)^2_{\pi/2\omega}. \tag{1.2}$$

In accordance with intuition the position distribution is contractive for $\omega>\omega'$. Only at the times $\omega t\in\frac{1}{2}\pi\mathbb{Z}$ is the Heisenberg relation saturated: $(\Delta P)_t(\Delta Q)_t=\hbar/2$. This is when the orbit passes through the set of quasiclassical vectors. Equations (1.1) and (1.2) demonstrate that the vector at time $t=\pi/2\omega$ is squeezed relative to the one at $t=0$ or, more specifically, it is coherent with respect to the harmonic oscillator Hamiltonian with frequency $\omega(\omega/\omega')$ and the same mass. After one half of the oscillator period, $t=\pi/\omega$, the orbit passes through the same set of coherent vectors it started from, but in general at a different point, as the position and momentum expectation values have the period $2\pi/\omega$. At $t=3\pi/2\omega$ the orbit passes through the same set of coherent vectors as at $t=\pi/2\omega$ and finally it closes at $t=2\pi/\omega$.

Section 2 contains the precise definitions of Gaussian and quasiclassical vectors and those of their properties which I need to compute their time evolution under a harmonic oscillator dynamics in § 4. The auxiliary material on Bogoliubov transformations is listed in § 3. (Sections 2 and 3 contain, perhaps modulo the degree of formal precision, mainly well established results and are intended as a concise repository of formulae relevant to §§ 4 and 5.) As an application of the harmonic time evolution of Gaussian vectors, the problem of exciting the higher levels of an oscillator by means of a perturbing frequency jump is analysed in § 5.

2. Gaussian vectors

Let a Hilbert space $\mathscr{H}$ with scalar product $(.,.)$ be given and let an irreducible representation of the canonical commutation relations (CCR) be operative in $\mathscr{H}$:

$$[P, Q] = -\mathrm{i}\hbar I. \tag{2.1}$$

For illustration the x-space representation of the CCR will be used:

$$\mathscr{H} := L^2(\mathbb{R}) \qquad (Q\Phi)(x) := x\Phi(x) \qquad (P\Phi)(x) := -\mathrm{i}\hbar\{\mathrm{d}\Phi/\mathrm{d}x\}(x). \tag{2.2}$$

Definition 2.1. A vector $\Phi \in \mathscr{H}$ is called Gaussian with respect to a representation P and Q of the CCR$\Leftrightarrow \exists (p, q) \in \mathbb{R}^2, \exists \lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$ such that

$$\Phi = \Psi[p, q, \lambda] := \exp[-\mathrm{i}(qP - pQ)/\hbar]\Psi[0, 0, \lambda] \tag{2.3}$$

$$\left(Q + \mathrm{i}\frac{\lambda}{\hbar}P\right)\Psi[0, 0, \lambda] = 0 \qquad \|\Psi[0, 0, \lambda]\| = 1 \tag{2.4}$$

holds. $G(P, Q) := \{\Psi[p, q, \lambda]: (p, q) \in \mathbb{R}^2, \lambda \in \mathbb{C}, \operatorname{Re} \lambda > 0\}$ denotes the set of vectors which are Gaussian relative to P and Q.

Notice that the vectors $\Psi[p, q, \lambda]$ are determined through definition 2.1 uniquely up to an unspecified complex factor of modulus equal to 1 and that $(p, q, \lambda) \neq (p', q', \lambda')$ implies $\Psi[p, q, \lambda] \neq \mathrm{e}^{\mathrm{i}\delta}\Psi[p', q', \lambda']$ for all δ in $[0, 2\pi)$. Thus the parameters p, q, λ, constitute a one-to-one parametrisation of the unit rays associated with $G(P, Q)$ in contrast to the one given in [6] for the same set of rays. The crucial property of the boost and translation operator which maps $\Psi[0, 0, \lambda]$ onto $\Psi[p, q, \lambda]$ is listed in lemma 2.1.

Lemma 2.1. $\forall (p, q) \in \mathbb{R}^2$ hold the equations:

$$\exp[\mathrm{i}(qP - pQ)/\hbar]P\exp[-\mathrm{i}(qP - pQ)/\hbar] = P + p \tag{2.5}$$

$$\exp[\mathrm{i}(qP - pQ)/\hbar]Q\exp[-\mathrm{i}(qP - pQ)/\hbar] = Q + q. \tag{2.6}$$

Example 2.1. The ground-state vector of the translated and boosted harmonic oscillator Hamiltonian $(P - p)^2/2m + m\omega^2(Q - q)^2/2$ is given by $\Psi(p, q, \lambda = \hbar/m\omega)$. This follows from lemma 2.1.

Example 2.2. Equation (2.7) displays the x-space representation of $\Psi[0, 0, \lambda]$:

$$\begin{aligned} &\Psi[0, 0, \lambda](x) = N(\lambda)\exp(-x^2/2\lambda) \\ &N(\lambda) := \mathrm{e}^{\mathrm{i}\delta}\left(\frac{1}{\pi}\operatorname{Re}(\lambda^{-1})\right)^{1/4} \end{aligned} \tag{2.7}$$

with $\delta \in [0, 2\pi)$ and the complex function $z^{1/4}$ cut along $0 > z \in \mathbb{R}$.

From proposition (2.1) the significance of the parameters, which label the Gaussian vectors, can be read off.

Proposition 2.1. $\forall (p, q) \in \mathbb{R}^2, \forall \lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$ hold the equations

$$\|\Psi[p, q, \lambda]\| = 1 \tag{2.8}$$

$$(\Psi[p, q, \lambda], P\Psi[p, q, \lambda]) = p \tag{2.9}$$

$$(\Psi[p, q, \lambda], Q\Psi[p, q, \lambda]) = q \tag{2.10}$$

$$(\Delta P)^2_{p,q,\lambda} := (\Psi[p, q, \lambda], P^2\Psi[p, q, \lambda]) - p^2 = \hbar^2/2 \operatorname{Re} \lambda \tag{2.11}$$

$$(\Delta Q)^2_{p,q,\lambda} := (\Psi[p, q, \lambda], Q^2\Psi[p, q, \lambda]) - q^2 = |\lambda|^2/2 \operatorname{Re} \lambda \tag{2.12}$$

$$(\Delta P)^2_{p,q,\lambda}(\Delta Q)^2_{p,q,\lambda} = \tfrac{1}{4}\hbar^2[1 + (\operatorname{Im} \lambda/\operatorname{Re} \lambda)^2] \geq \hbar^2/4. \tag{2.13}$$

Proof. The proof proceeds in full analogy to the well known case of real λ. Equation (2.8) follows from the unitarity of $\exp[-i(qP - pQ)/\hbar]$. Due to lemma 2.1 it is sufficient to verify the equations (2.9)-(2.11) in the case $p = q = 0$. Define $a(\lambda) := (Q + i\lambda P/\hbar)/\sqrt{2 \operatorname{Re} \lambda}$ and verify $P/\hbar = i[a(\lambda)^\dagger - a(\lambda)]/\sqrt{2 \operatorname{Re} \lambda}$, $Q = [\lambda a(\lambda)^\dagger + \lambda^* a(\lambda)]/\sqrt{2 \operatorname{Re} \lambda}$, $[a(\lambda), a(\lambda)^\dagger] = I$. $a(\lambda)\Psi[0, 0, \lambda] = 0$ then leads to the statement.

Definition 2.2. $\Phi \in \mathscr{H}$ is called a quasiclassical (or minimum uncertainty) vector with respect to the representation P and Q of the CCR$\Leftrightarrow$ $(\Delta P)_\Phi(\Delta Q)_\Phi = \hbar/2$ with $(\Delta X)^2_\Phi := \{(\Phi, X^2\Phi) - (\Phi, X\Phi)^2\}$ and $\|\Phi\|^2 = 1$. $M(P, Q) := \{\Phi \in \mathscr{H}: (\Delta P)_\Phi(\Delta Q)_\Phi = \hbar/2, \|\Phi\|^2 = 1\}$ denotes the set of vectors which are quasiclassical with respect to P and Q.

Proposition 2.2. $M(P, Q) \subset G(P, Q)$ and more specifically the following criterion holds $\forall \Psi[p, q, \lambda] \in G(P, Q)$:

$$\Psi[p, q, \lambda] \in M(P, Q) \Leftrightarrow \operatorname{Im} \lambda = 0.$$

Proof. The proof of $M(P, Q) \subset G(P, Q)$ can be found in [4]. The remaining criterion is an immediate consequence of equation (2.13).

Proposition (4.1) gives relevance to the definition of 'coherent' vectors due to their particularly simple evolution under the dynamics $P^2/2m + m\omega^2Q^2/2$.

Definition 2.3. Let $m \in \mathbb{R}$, $m > 0$, $\omega \in \mathbb{R}$, $\omega > 0$. The set $C(P, Q; \hbar/m\omega) := \{\Psi[p, q, \lambda = \hbar/m\omega]: (p, q) \in \mathbb{R}^2\} \subset M(P, Q)$ is called the set of coherent vectors with respect to the representation P and Q of the CCR and $P^2/2m + m\omega^2Q^2/2$.

Note that $M(P, Q)$ is obtained by forming the union of all sets of coherent vectors, i.e. that $\bigcup_{\lambda \in \mathbb{R}_+} C(P, Q; \lambda) = M(P, Q)$ holds.

3. Bogoliubov transformations

The notion of Bogoliubov transformations is usually introduced via the raising and lowering operators associated with the harmonic oscillator Hamiltonian.

Definition 3.1. $\forall \eta \in \mathbb{R}$, $\eta > 0$ define the dimensionless operators:

$$\hat{P}_\eta := \eta P/\hbar \qquad \hat{Q}_\eta := Q/\eta \qquad a(\eta) := (\hat{Q}_\eta + i\hat{P}_\eta)/\sqrt{2}.$$

Lemma 3.1. $\forall \eta \in \mathbb{R}$, $\eta > 0$ holds $[\hat{P}_\eta, \hat{Q}_\eta] = -\mathrm{i}I$ and $[a(\eta), a(\eta)^\dagger] = I$.

The raising and lowering operators, which are associated with the harmonic oscillator Hamiltonian (with the parameters m and ω), are obtained by putting $\eta = (\hbar/m\omega)^{1/2}$, since $\forall m \in \mathbb{R}$, $m > 0$, $\forall \omega \in \mathbb{R}$, $\omega > 0$ holds:

$$P^2/2m + m\omega^2 Q^2/2 = \hbar\omega\{a[\hbar/m\omega]^{1/2\dagger} a[\hbar/m\omega]^{1/2} + I/2\}. \tag{3.1}$$

Definition 3.2. $\forall \eta \in \mathbb{R}$, $\eta > 0$, $\forall (s, t) \in \mathbb{C}^2$ with $|s|^2 - |t|^2 = 1$ the definition $a(\eta; s, t) := sa(\eta) + ta(\eta)^\dagger$ is called a Bogoliubov transformation.

Lemma 3.2. $\forall \eta \in \mathbb{R}$, $\eta > 0$ and $\forall (s, t) \in \mathbb{C}^2$ with $|s|^2 - |t|^2 = 1$ holds:

$$[a(\eta; s, t), a(\eta; s, t)^\dagger] = I. \tag{3.2}$$

Proposition 3.1. $\forall \eta \in \mathbb{R}$, $\eta > 0$ and $\forall (s, t) \in \mathbb{C}^2$ with $|s|^2 - |t|^2 = 1$ the following equivalence holds:

$$\Phi \in \mathscr{H} \qquad \|\Phi\| = 1 \qquad a(\eta; s, t)\Phi = 0 \qquad \Leftrightarrow \qquad \exists \delta \in [0, 2\pi)$$

with

$$\Phi = \mathrm{e}^{\mathrm{i}\delta}\Psi\left(0, 0, \lambda = \frac{s-t}{s+t}\eta^2\right).$$

Proof. Expressing $a(\eta; s, t)$ through P and Q leads to

$$a(\eta; s, t) = \left(Q + \mathrm{i}\frac{s-t}{s+t}\eta^2 P/\hbar\right)(s+t)/\eta\sqrt{2}$$

which implies the statement.

Definition 3.3. $\forall z \in \mathbb{C}$, $\forall \alpha \in \mathbb{R}$, $\forall \eta \in \mathbb{R}$, $\eta > 0$ define the unitary operator $B_\eta(\alpha, z)$ on $\mathscr{H}$:

$$B_\eta(\alpha, z) := \exp[-\mathrm{i}\alpha a(\eta)^\dagger a(\eta)] \exp\{-\tfrac{1}{2}\mathrm{i}[za(\eta)^\dagger a(\eta)^\dagger + z^* a(\eta)a(\eta)]\}. \tag{3.3}$$

Proposition 3.2 demonstrates that the operators $B_\eta(\alpha, z)$ implement the Bogoliubov transformations unitarily.

Proposition 3.2 [4]. $\forall z \in \mathbb{C}$, $\forall \alpha \in \mathbb{R}$, $\forall \eta \in \mathbb{R}$, $\eta > 0$ holds:

$$B_\eta(\alpha, z)^\dagger = B_\eta(\alpha, z)^{-1} \qquad \text{(unitarity)} \tag{3.4}$$

$$B_\eta(\alpha, z)^\dagger a(\eta) B_\eta(\alpha, z) = s(\alpha, z)a(\eta) + t(\alpha, z)a(\eta)^\dagger \tag{3.5}$$

$$s(\alpha, z) := \mathrm{e}^{-\mathrm{i}\alpha}\cosh|z| \qquad t(\alpha, z) := \mathrm{e}^{-\mathrm{i}\alpha}\left(-\mathrm{i}\frac{z}{|z|}\right)\sinh|z|. \tag{3.6}$$

Proof. Unitarity is obvious. The transformation law (3.5) can be verified by computing $B_\eta(\alpha, z)^\dagger a(\eta) B_\eta(\alpha, z)$ in two steps according to the decomposition of $B_\eta(\alpha, z)$: $B_\eta(\alpha, z) = B_\eta(\alpha, 0)B_\eta(0, z)$. First the α dependence is obtained by observing

$$B_\eta(\alpha, 0)^\dagger a(\eta) B_\eta(\alpha, 0) = \mathrm{e}^{-\mathrm{i}\alpha} a(\eta).$$

(This equation, familiar from a harmonic oscillator's time evolution, follows by solving the first-order ordinary differential equation, which is obtained by differentiating $B_\eta(\alpha, 0)^\dagger a(\eta) B_\eta(\alpha, 0)$ with respect to α and employing the CCR for $a(\eta)$ from lemma 3.1.) Then in a second step the z dependence of the resulting expression $\mathrm{e}^{-\mathrm{i}\alpha} B_\eta(0, z)^\dagger a(\eta) B_\eta(0, z)$ can be obtained by solving the first-order ordinary differential equation, which results through differentiation of $B_\eta(0, \lambda z)^\dagger a(\eta) B_\eta(0, \lambda z)$ with respect to the real-valued auxiliary variable λ for arbitrary z. The differential equation's solution is uniquely determined by its initial value $a(\eta)$ at $\lambda = 0$, which is obvious from $B_\eta(0, 0) = I$.

Note that the mapping $[0, 2\pi)\times\mathbb{C}\to\{(s,t)\in\mathbb{C}: |s|^2-|t|^2=1\}$, which operates as $(\alpha, z)\mapsto(s(\alpha,z), t(\alpha,z))$, is one to one. Thus to each pair $(s,t)\in\mathbb{C}^2$, with $|s|^2-|t|^2=1$, there corresponds exactly one element B_η which implements the Bogoliubov transformation of definition 3.2. Note also that the set of all Bogoliubov unitaries with fixed η, namely $\{B_\eta(\alpha, z): \alpha\in[0,2\pi), z\in\mathbb{C}\}$, is independent of η and that a change of η only amounts to a reparametrisation of this group of operators.

Example 3.1 illustrates the Bogoliubov transformations through the interrelation between the raising and lowering operators of different mass and frequency parameters.

Example 3.1. $\forall\eta\in\mathbb{R}, \eta>0, \forall\eta'\in\mathbb{R}, \eta'>0$ holds:

$$a(\eta')=s(\eta/\eta')a(\eta)+t(\eta/\eta')a(\eta)^\dagger$$
$$=B_\eta(0,\tfrac{1}{2}\mathrm{i}\ln(\eta'/\eta))^\dagger a(\eta)B_\eta(0,\tfrac{1}{2}\mathrm{i}\ln(\eta'/\eta)) \tag{3.7}$$
$$s(\eta/\eta'):=(1+\eta/\eta')/2(\eta/\eta')^{1/2} \tag{3.8}$$
$$t(\eta/\eta'):=(1-\eta/\eta')/2(\eta/\eta')^{1/2}. \tag{3.9}$$

Note that $\forall x\in\mathbb{R}, x>0$ the relation $|s(x)|^2-|t(x)|^2=1$ is true.

Proposition 3.3. Let U be one of the operators $B_\eta(\alpha, z)$ from definition 3.3. Then the following equivalence holds $\forall\eta\in\mathbb{R}, \eta>0$:

$$U^\dagger a(\eta)U=sa(\eta)+ta(\eta)^\dagger \Leftrightarrow U^\dagger PU=\frac{\hbar}{\eta^2}\mathrm{Im}(s-t^*)Q+\mathrm{Re}(s-t^*)P \tag{3.10}$$

and

$$U^\dagger QU=\mathrm{Re}(s+t^*)Q-\mathrm{Im}(s+t^*)\eta^2P/\hbar. \tag{3.11}$$

Proof. Express P and Q through $a(\eta)$ and $a(\eta)^\dagger$.

From proposition 3.3 the set of squeezing Bogoliubov transformations of [3] is recovered by specialising to $s\in\mathbb{R}$, $t\in\mathbb{R}$ and $s-t=:\mu>0$. Note that $s+t=\mu^{-1}$ holds. Proposition 3.2, together with definition 3.3, implies that the squeezing Bogoliubov transformations are the ones with $\alpha=0$ and $\mathrm{Re}(z)=0$. This displays lucidly that a harmonic oscillator Hamiltonian does not generate these transformations.

4. Harmonic time evolution of Gaussian vectors

Let $\{H(t)\}_{t\in\mathbb{R}}$ be a one-parameter family of self-adjoint Hamiltonian operators with the associated family of unitary time evolution operators $U(t)$ defined through

$$\mathrm{i}\hbar\frac{\partial}{\partial t}U(t)=H(t)U(t) \qquad U(0)=I. \tag{4.1}$$

Definition 4.1. A time evolution operator $U(t)$ at a fixed time is said to squeeze a vector $\Phi\in M(P,Q)$ with respect to Q (or P) $\Leftrightarrow U(t)\Phi\in M(P,Q)$ and $(\Delta Q)_{U(t)\Phi}<(\Delta Q)_\Phi$ (or $(\Delta P)_{U(t)\Phi}<(\Delta P)_\Phi$).

Definition 4.2. The harmonic oscillator Hamiltonian with real frequency and mass parameters $\omega \geqslant 0$, $m>0$:

$$H(\omega) := P^2/2m + m\omega^2 Q^2/2 \tag{4.2}$$

$$U_\omega(t) := \exp[-\mathrm{i}H(\omega)t/\hbar]. \tag{4.3}$$

Proposition 4.1 displays the orbits which are generated by the harmonic oscillator time evolution in the set of Gaussian vectors $G(P, Q)$.

Proposition 4.1. $\forall t \in \mathbb{R}$, $\forall \omega \in \mathbb{R}$, $\omega \geqslant 0$ holds:

$$\begin{aligned} &U_\omega(t)G(P,Q) = G(P,Q) \\ &U_\omega(t)\Psi[p,q,\lambda] = \exp[\mathrm{i}\delta(\omega,t)]\Psi[p_\omega(t), q_\omega(t), \lambda_\omega(t)] \end{aligned} \tag{4.4}$$

with $\delta(\omega, t) \in [0, 2\pi)$ and

$$p_\omega(t) := p\cos(\omega t) - m\omega q \sin(\omega t) \tag{4.5}$$

$$q_\omega(t) := q\cos(\omega t) + p\sin(\omega t)/m\omega \tag{4.6}$$

$$\lambda_\omega(t) := \frac{\mathrm{Re}\,\lambda + \mathrm{i}\{\mathrm{Im}\,\lambda \cos(2\omega t) + [1-(m\omega/\hbar)^2|\lambda|^2](\hbar/2m\omega)\sin(2\omega t)\}}{[\cos(\omega t) - (m\omega/\hbar)\,\mathrm{Im}\,\lambda \sin(\omega t)]^2 + (m\omega/\hbar)^2(\mathrm{Re}\,\lambda)^2 \sin^2(\omega t)}. \tag{4.7}$$

Proof. The time dependence of $p_\omega(t)$ and $q_\omega(t)$ follows from commuting $U_\omega(t)$ through $\exp[-\mathrm{i}(qP - pQ)/\hbar]$. This can be done easily by using the solution of Heisenberg's operator equations of motion, which can be extracted from proposition 3.3 by putting $z=0$. $\lambda_\omega(t)$ can be derived by acting on the defining equation (2.4) for $\Psi[0,0,\lambda]$ with $U_\omega(t)$. Commuting $U_\omega(t)$ through $Q + \mathrm{i}\lambda P/\hbar$ again can be done with help of the solution of Heisenberg's operator equations of motion and leads to the stated result.

Observe that $\lambda_\omega(0) = \lambda$, $p_\omega(0) = p$, $q_\omega(0) = q$ indeed holds. The limiting case $\omega \downarrow 0$ reproduces the free Gaussian wavepacket correctly: $\lambda_0(t) = \lambda + \mathrm{i}t\hbar/m$, $p_0(t) = p$, $q_0(t) = q + pt/m$, such that, according to proposition 2.1, the position variance is given by $(\Delta Q)_t^2 = [(\mathrm{Re}\,\lambda)^2 + (\mathrm{Im}\,\lambda + t\hbar/m)^2]/2\,\mathrm{Re}\,\lambda$. The free wavepacket contracts before $t = -m\,\mathrm{Im}\,\lambda/\hbar$ and spreads afterwards. The packet is quasiclassical at $t = -m\,\mathrm{Im}\,\lambda/\hbar$ only. For $\mathrm{Im}\,\lambda < 0$, which can always be made true by a shift of the timescale, these orbits have been named 'contractive' and have been employed in the modelling of position measurements for a free quantum [6].

Further notice that $\lambda_\omega(t)$ is insensitive to p and q. One may say that the internal motion of the Gaussian wavepacket decouples from the external one. That $\lambda_\omega(t)$ is periodic in t with half of the oscillator's period deserves some comment. The parity conservation of $U_\omega(t)$ and the fact that $\Psi[0,0,\lambda]$ has even parity, as is obvious from example 2.2, implies that only even-parity eigenvectors of $H(\omega)$, corresponding to the eigenvalues $[\hbar\omega(2n+\frac{1}{2})]_{n\in\mathbb{N}_0}$, yield non-zero scalar products with $\Psi[0,0,\lambda]$. Thus only these eigenvectors contribute to an expansion of $\Psi[0,0,\lambda]$ into the eigenbasis of $H(\omega)$, which explains the period of λ_ω.

From the formula for $(\Delta P)_{p,q,\lambda}(\Delta Q)_{p,q,\lambda}$ given by proposition 2.1 and the one for $\lambda_\omega(t)$ in proposition 4.1 the times at which $\Psi[p_\omega(t), q_\omega(t), \lambda_\omega(t)]$ is quasiclassical can be read off to be as stated in corollary 4.1.

Corollary 4.1. $\forall (p,q) \in \mathbb{R}^2$, $\forall \lambda \in \mathbb{C}$, $\mathrm{Re}\,\lambda > 0$ holds the equivalence:

$$U_\omega(t)\Psi[p,q,\lambda] \in M(P,Q) \Leftrightarrow \mathrm{Im}\,\lambda\cos(2\omega t) + [1-(m\omega/\hbar)^2|\lambda|^2]\frac{\hbar}{2m\omega}\sin(2\omega t) = 0.$$

Three exhaustive cases of initial conditions need a separate discussion. First the trivial case $\lambda = \hbar/m\omega$, which corresponds to choosing a coherent vector with respect to $H(\omega)$ as the initial condition at $t=0$. In this case follows, $\forall t \in \mathbb{R}$:

$$U_\omega(t)\Psi\left[p, q, \frac{\hbar}{m\omega}\right] \in C\left(P, Q, \frac{\hbar}{m\omega}\right).$$

The second case is $\operatorname{Im}\lambda \neq 0$ (and therefore $\lambda \neq \hbar/m\omega$), which corresponds to choosing a non-quasiclassical vector as initial condition. Here in the half open t-interval $[0, 2\pi/\omega)$ four and only four instants t of time exist with $\tan(2\omega t) = -\operatorname{Im}\lambda/\{[1-(m\omega/\hbar)^2|\lambda|^2]\hbar/2m\omega\}$ such that $U_\omega(t)\Psi[p, q, \lambda] \in M(P, Q)$ follows. Finally, there is the case $\lambda \neq \hbar/m\omega$ with $\operatorname{Im}\lambda = 0$, which corresponds to choosing a coherent vector with respect to $H(\omega')$, with $\omega \neq \omega'$, as the (quasiclassical) initial condition, and into which case 2 can be transformed by shifting the timescale and determining ω' properly in terms of λ and ω. This last case, therefore, is the general generic case. Here in the t interval $[0, 2\pi/\omega)$ again exactly four instants t of time exist, with $\sin(2\omega t) = 0$, such that $U_\omega(t)\Psi[p, q, \lambda] \in M(P, Q)$ follows. These times are $\omega t \in \{0, \pi/2, \pi, 3\pi/2\}$. The position and momentum variances for this case, to be read off from propositions 2.1 and 4.1, are listed in example 4.1. A simple alternative derivation of example 4.1 can be drawn from the solutions of Heisenberg's operator equations of motion for $U_\omega(t)^\dagger P U_\omega(t)$ and $U_\omega(t)^\dagger Q U_\omega(t)$ as they are implicit in proposition 3.3.

Example 4.1. $\forall \hbar/m\omega' \in \mathbb{R}$, $0 < \hbar/m\omega'$, $\forall (p, q) \in \mathbb{R}^2$, $\forall \omega \in \mathbb{R}$, $0 < \omega$ holds:

$$(\Delta P)^2_{U_\omega(t)\Psi[p, q, \hbar/m\omega']} = \tfrac{1}{2}\hbar m\omega'\{1 + [(\omega/\omega')^2 - 1]\sin^2(\omega t)\} \tag{4.8}$$

$$(\Delta Q)^2_{U_\omega(t)\Psi[p, q, \hbar/m\omega']} = \frac{\hbar}{2m\omega'}\{1 + [(\omega'/\omega)^2 - 1]\sin^2(\omega t)\}. \tag{4.9}$$

Note that for $\omega' < \omega$ the position variance decreases below its starting value of $(\Delta Q)^2_{\max} = \hbar/2m\omega'$ at $t=0$, which is also its maximum value, to attain its minimum value of

$$(\Delta Q)^2_{\min} = \frac{\hbar}{2m\omega'}\left(\frac{\omega'}{\omega}\right)^2 < \frac{\hbar}{2m\omega'}$$

at time $\omega t = \pi/2$ before it returns to the starting value at $\omega t = \pi$ and the process repeats periodically. Note that if and only if the position variance is in its minimum or maximum, then $(\Delta P)(\Delta Q) = \hbar/2$ holds. Therefore the harmonic oscillator dynamics with frequency ω evolves the coherent vectors with respect to the harmonic oscillator Hamiltonian with smaller frequency parameter $\omega' < \omega$ into relatively Q-squeezed ones at the times $\omega t = (2n+1)\pi/2$ with $n \in \mathbb{Z}$.

5. Harmonic oscillator with a frequency jump

Finally, the question of how the orbits of proposition 4.1 could possibly be realised is touched upon in this section. Consider a harmonic oscillator Hamiltonian with the frequency being a function of time, which jumps at $t=0$ from ω' to ω, then at time $T>0$ back to ω', and is constant otherwise. The dynamical orbits induced by this time-dependent Hamiltonian are of the type considered in example 4.1. If the ground-state vector $\Psi[0, 0, \hbar/m\omega']$ of $H(\omega')$ is assumed to be realised at $t=0$, then this vector (up to a phase) is realised for all negative times, while for positive times t between 0 and T the vector (up to a phase) is given by $\Psi[0, 0, \lambda_\omega(t)]$ with the starting value $\lambda = \hbar/m\omega'$. For $t \geq T$ the curve is given (up to a phase) by $\Psi[0, 0, \lambda_{\omega'}(t-T)]$ with the starting value $\lambda_{\omega'}(0) = \lambda_\omega(T)$. For $t \in [0, T]$ the momentum and position variances are as stated in example 4.1. An important observable quantity of $\Psi[0, 0, \lambda_\omega(T)]$, namely the occupation probability of the excited levels of $H(\omega')$:

$$P_{\omega'/\omega}(\omega T) := 1 - |(\Psi(0, 0, \lambda = \hbar/m\omega'), \Psi[0, 0, \lambda_\omega(T)])|^2 \tag{5.1}$$

as a function of the perturbation's duration T, can be read off from proposition 5.2.

Proposition 5.2. $\forall m \in \mathbb{R}$, $m > 0$, $\forall \omega' \in \mathbb{R}$, $\omega' > 0$, $\forall \omega \in \mathbb{R}$, $\omega > 0$, $\forall T \in \mathbb{R}$, with $\tau := \omega T$, $\lambda := \hbar/m\omega'$ and $\lambda_\omega(T)$ as defined in proposition 4.1, holds:

$$|(\Psi(0, 0, \lambda = \hbar/m\omega'), \Psi[0, 0, \lambda_\omega(T)])|^2 = [1 + \tfrac{1}{4}(\omega'/\omega - \omega/\omega')^2 \sin^2 \tau]^{-1/2}. \tag{5.2}$$

Proof. Use the x-space representation of $\Psi[0, 0, \lambda]$ to compute the scalar product via integration. This yields equation (5.3):

$$|(\Psi(0, 0, \lambda = \hbar/m\omega'), \Psi[0, 0, \lambda_\omega(T)])|^2$$
$$= 2\left(\frac{[\cos^2 \tau + (\omega'/\omega)^2 \sin^2 \tau]}{(1 + \cos^2 \tau + (\omega'/\omega)^2 \sin^2 \tau)^2 + \cos^2 \tau \sin^2 \tau (\omega'/\omega - \omega/\omega')^2}\right)^{1/2}. \tag{5.3}$$

The denominator can be further simplified as follows:

$$(1 + \cos^2 \tau + (\omega'/\omega)^2 \sin^2 \tau)^2 + \cos^2 \tau \sin^2 \tau \left(\frac{\omega'}{\omega} - \frac{\omega}{\omega'}\right)^2$$
$$= 1 + 2[\cos^2 \tau + (\omega'/\omega)^2 \sin^2 \tau] + [\cos^2 \tau + (\omega'/\omega)^2 \sin^2 \tau]^2$$
$$+ \cos^2 \tau \sin^2 \tau \left(\frac{\omega'}{\omega} - \frac{\omega}{\omega'}\right)^2.$$

Check now the relation

$$1 + \cos^2 \tau \sin^2 \tau \left(\frac{\omega'}{\omega} - \frac{\omega}{\omega'}\right)^2 = [\cos^2 \tau + (\omega'/\omega)^2 \sin^2 \tau][\cos^2 \tau + (\omega/\omega')^2 \sin^2 \tau]$$

which implies that the denominator can be written as

$$[\cos^2 \tau + (\omega'/\omega)^2 \sin^2 \tau]\{2 + [\cos^2 \tau + (\omega'/\omega)^2 \sin^2 \tau] + [\cos^2 \tau + (\omega/\omega')^2 \sin^2 \tau]\}.$$

From this then equation (5.2) follows immediately.

Note that the excitation probability is 0 for $T = 0$ and increases to reach its maximum value of $1 - 2(\omega/\omega' + \omega'/\omega)^{-1} \geq 0$ at $\omega T = \pi/2$, then returns to the value 0 at $\omega T = \pi$ and repeats this process periodically. Observe also the surprising symmetry: $P_{\omega'/\omega}(\tau) = P_{\omega/\omega'}(\tau)$.

As expected intuitively, the energy transfer into the system through an instantaneous frequency jump at $t = 0$ is positive if the oscillator frequency is increased (negative if the frequency drops). It is given by

$$E(t \in [0, T]) - E(t \in (-\infty, 0])$$
$$:= (\Psi[0, 0, \hbar/m\omega'], \{H(\omega) - H(\omega')\}\Psi[0, 0, \hbar/m\omega'])$$
$$= \hbar\omega' \left(\frac{\omega^2 - \omega'^2}{4\omega'^2}\right). \tag{5.4}$$

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Note added. A similar attitude towards [3] as presented in this paper seems to be adopted by [9].

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